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Effects of helicity in strongly anisotropic turbulence

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Effect of helicity in anisotropic turbulence is studied, by using the energy and helicity spectrum equations that are closed by the EDQNM theory, assuming that the turbulent field is homogeneous and axisymmetric.

I. INTRODUCTION

In previous papers,[1,2] we have studied two-dimensionality in low magnetic Reynolds number turbulence with non helicity under a uniform magnetic field $\mathbf{B}_0$ and randomly stirred forces, assuming that the turbulent field is homogeneous and axisymmetric. We have shown that when $\mathbf{B}_0$ is strong, the MHD turbulence exhibits properties of two-dimensional turbulence such as inverse cascade with $k^{-5/3}$ energy spectrum and enstrophy cascade towards high wavenumber with k^{-3} energy spectrum.

In the present paper, following the previous papers,[1,2] we study the effects of helicity in low magnetic Reynolds number MHD turbulence. The flows of such low magnetic Reynolds number fluid are characterized by a magnetic diffusion time much smaller than any other time scale. The dynamics of such flows is described by so-called quasi-static approximation: [3,4] that is,

$$\frac{\partial \hat{\mathbf{u}}(\mathbf{x},t)}{\partial t} + (\hat{\mathbf{u}}.\nabla)\hat{\mathbf{u}} = -\frac{1}{\rho}\nabla \hat{p} + \nu\Delta\hat{\mathbf{u}} + \hat{\mathbf{F}}(\mathbf{x},t)\ , \tag{1}$$

$$\nabla.\hat{\mathbf{u}} = 0\ , \tag{2}$$

$$\mathbf{F}(\mathbf{k},t) = -\frac{\sigma B_0^2}{\rho}\cos^2\theta\ \mathbf{u}(\mathbf{k},t) \tag{3}$$

where $\mathbf{u}(\mathbf{k},t)$ and $\mathbf{F}(\mathbf{k},t)$ are the Fourier transforms of the electromagnetic force $\hat{\mathbf{F}}(\mathbf{x},t)$ and the turbulent velocity $\hat{\mathbf{u}}(\mathbf{x},t)$, respectively: ν, ρ, and σ are the kinematic viscosity, uniform fluid density and electric conductivity, respectively: θ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$. The electromagnetic effect then reduces to an anisotropic Joule dissipation with a rate proportional to the squared cosine of the angle θ between the wave vector $\mathbf{k}$ and the uniform external magnetic field $\mathbf{B}_0$, but independent of wavenumber. The problem is then closer to Navier-Stokes fluids rather than a MHD problem.

Turbulence with net helicity has received much attention because of its importance in the generation of turbulent magnetic fields.[5,6] Such turbulence is also of basic theoretical importance because it emphasizes constants of motion. Moffatt[7] showed that the total helicity of a bounded flow, defined as

$$\int (\hat{\mathbf{u}}.\hat{\boldsymbol{\omega}})\ d^3x \tag{4}$$

is an inviscid constant of motion.

The two quadratic constants of motion, kinetic energy and helicity, in inviscid three-dimensional flow suggest that there may be some analogies between three-dimensional turbulence with non-zero helicity and two-dimensional turbulence, where there are also two quadratic constants: energy and enstrophy. Kraichnan[8] explored the analogy between the two kinds of flow, examining the absolute inviscid equilibrium ensembles with net helicity, following his earlier treatment of absolute equilibrium ensembles in two-dimensional flow.[9] He pointed out that the analogy is not a close one, despite the existence of two quadratic constants of motion in each case. The absolute equilibrium ensembles for the helical turbulence show none of the interesting structure associated with negative temperatures in the two-dimensional system. Kraichnan's conclusion was that the inertial range cascade of energy in isotropic turbulence should not differ qualitatively from that in ordinary reflexion-invariant turbulence. The major new qualitative effect with strongly helical turbulence promises to be the inhibited flow to lower wave numbers. This is not important in isotropic flows, but may be important in other situations.

As mentioned above, helicity is important for the generation of magnetic fields in MHD flows. In neutral flows, helicity seems to be characterized by a strong inhibition of the energy transfers towards small scales. This point could have implications in meteorogy, where it has been suggested that helicity contained in the small scale atmospheric motions (tornados for instance) could increase the large scale atmospheric predictability.[10,11] Such an inhibition of the transfers can be justified in various ways: a very simple argument on the basis of the vorticity equation

$$\frac{\partial \hat{\boldsymbol{\omega}}}{\partial t} + \nabla \times (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{u}}) = \nu \nabla^2 \hat{\boldsymbol{\omega}} \tag{5}$$

says that in the non helical case, and since $\langle \hat{\boldsymbol{\omega}} \cdot \hat{\mathbf{u}} \rangle = 0$ the vorticity is a perpendicular "in the mean" to the velocity, and then the vectorial products $\langle \hat{\boldsymbol{\omega}} \times \hat{\mathbf{u}} \rangle$ and the energy transfers are maximum.

This tendency can also be quantified with the aid of a freely-evolving EDQNM calculation, starting with the energy spectrum sharply peaked in the vicinity of k_i, with the maximum helicity

$$H(k,t{=}0) = kE(k,t{=}0) \tag{6}$$

permitted by inequality: these calculations, presented by Andre *et al.*[12], show without ambiguity the following results:

i) the critical time t* at which the enstrophy blows up (in the inviscid limit) is increased by nearly a factor 2, and is now of the order of $9/v_0 k_i(0)$. The "enstrophy of the helicity" $\int_0^\infty k^2 H(k,t)\mathrm{d}k$ blows up at the same time. The kinetic energy starts being dissipated at a finite rate at that time, and so does the helicity.

ii) at t* appear simultaneously $k^{-5/3}$ inertial ranges for the energy spectrum and the helicity spectrum. The helicity spectrum follows a "linear" cascade

$$H(k) \approx \frac{\varepsilon_H}{\varepsilon} E(k) = 2.25 \varepsilon_H \varepsilon^{-1/3} k^{-5/3} . \tag{7}$$

iii) the "relative" helicity $H(k)/kE(k)$ is, in the inertial-range, proportional to k^{-1}, and decreases rapidly with k. Then the helicity has not real influence on the energy flux, expressed in terms of the energy dissipation rate. It follows that Kolmogorov constant in the energy cascade is not modified by the presence of helicity.

New ideas were proposed by Tsinber[13] and Moffatt[14] where there would exist in a flow with zero mean helicity local regions (in the **x** space) with non zero helicity (positive or negative) where the kinetic energy dissipation would be less active than in the non helical regions, because of the preceding results concerning the inhibition of kinetic energy dissipation by helicity. The flow would then evolve towards a set of "coherent" helical structures separated by non helical dissipative structures (may be fractal). The coherent structures of same sign could possibly pair, leading to the inverse transfer sought. Up to now this is nothing more than a conjecture which has to be verified, with the aid of direct numerical simulations of turbulence for instance. Remark that an investigation of the transfers within the complex helical waves decomposition has not shown any tendency to significant positive transfer between helical waves of same polarization.[15] But it is still an isotropic study, and anisotropy could play a role in the development of these large structures.

II. ABSOLUTE EQUILIBRIUM ENSEMBLE IN QUASI TWO-DIMENSIONAL TURBULENCE

Let us consider the motion in the limit of two-dimensional flows.[11] We assume that the z axis of coordinates is directed along $\mathbf{B}_0$, and look for two-dimensional solutions $\mathbf{u}(x,y,t)$ and $P(x,y,t)$. Let $u(x,y,t)$ and $v(x,y,t)$ and $w(x,y,t)$ be respectively the "horizontal" (that is perpendicular to $\mathbf{B}_0$) and vertical components of the velocity. The continuity equation implies that the velocity field is horizontally non divergent

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{8}$$

The result is that the "horizontal" velocity field $\mathbf{u}_H(x,y,t)$ of components $(u,v,0)$ satisfies a two dimensional Navier-Stokes equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \nu \nabla_H^2 u, \tag{9}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \nu \nabla_H^2 v \tag{10}$$

with the incompressibility condition (8). In the following, the suffix H refers to the horizontal coordinates: for instance ∇_H^2 is the horizontal Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$. The vertical component $w(x,y,t)$ obeys a two-dimensional passive scalar equation

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\nu\nabla_H^2 w\ . \tag{11}$$

This shows that the assumption of two-dimensionality does not imply a purely horizontal motion: a fluid particle will conserve during the motion its initial velocity.

Following Kraichnan[8,9] and Frisch[16] we examine the absolute statistical equilibrium of the equations for truncated quasi-two-dimensional turbulence. They are obtained by considering solutions which, being periodic in space, are represented by Fourier series and then retaining only those Fourier components $\mathbf{v}(\mathbf{k})$ of the velocity field for which $|\mathbf{k}| = k \in \mathrm{I} = (k_{min}, k_{max})$, and dropping any nonlinear interaction terms involving one or several wave numbers outside this interval I. From (8)-(11), we can obtain 4 quadratic-invariants as

Total energy of transverse velocity components $E_n=\frac{1}{2}\sum_k \left(v_1(\mathbf{k})v_1^*(\mathbf{k})+v_2(\mathbf{k})v_2^*(\mathbf{k})\right)$, (12)

Total energy of parallel velocity components $E_p=\frac{1}{2}\sum_k v_3(\mathbf{k})v_3^*(\mathbf{k})$, (13)

Total helicity per unit volume $H=\frac{i}{2}\sum_k \varepsilon_{rmn}k_n v_r(\mathbf{k})v_m^*(\mathbf{k})$ (14)

Enstrophy $S=\frac{1}{2}\sum_k k^2\left(v_1(\mathbf{k})v_1^*(\mathbf{k})+v_2(\mathbf{k})v_2^*(\mathbf{k})\right)$ (15)

Notice that E_n, E_p, H and S are not mean values of invariants but the stochastic values corresponding to individual realizations.

The truncated equations of motion posses canonical equilibrium ensembles for which the density in phase space is

$$\mathrm{P_r}=\frac{1}{Z}\exp(-\alpha_n E_n-\alpha_p E_p-\beta H-\xi S) \tag{16}$$

where Z is a normalizing constant. This density is the exponential of a quadratic form and therefore defines a Gaussian ensemble. To calculate the second-order moments of the components of the velocity field the following lemma is used.

LEMMA. The multivariate Gaussian density

$$\rho(\mathbf{x})=\frac{1}{Z}\exp\left(-\frac{1}{2}\sum_{i,j} A_{ij}x_i x_j\right) \tag{17}$$

has second-order moments given by

$$\langle x_i x_j\rangle = A_{ij}^{-1} \tag{18}$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.

This result is obvious when $\mathbf{A}$ is diagonal and the general case is reducible to this special case by a linear orthogonal transformation.

To apply this result we must determine, for each wave vector $\mathbf{k}$, the real and imaginary parts of the two components of $\mathbf{v}(\mathbf{k})$ in a plane perpendicular to $\mathbf{k}$ (since $\mathbf{k}.\mathbf{v}(\mathbf{k}) = 0$) . Clearly the matrix **A** separates into 4×4 blocks corresponding to the different wave vectors. Each such block reads

$$\mathbf{A}(\mathbf{k}) = \begin{bmatrix} \alpha_n+\xi k^2 & 0 & 0 & -\beta k \\ 0 & \alpha_n+\xi k^2 & \beta k & 0 \\ 0 & \beta k & \alpha_p & 0 \\ -\beta k & 0 & 0 & \alpha_p \end{bmatrix}, \tag{19}$$

$$\mathbf{A}^{-1}(\mathbf{k}) = \frac{-1}{\Delta}\begin{bmatrix} -\alpha_p & 0 & 0 & -\beta k \\ 0 & -\alpha_p & \beta k & 0 \\ 0 & \beta k & -(\alpha_n+\xi k^2) & 0 \\ -\beta k & 0 & 0 & -(\alpha_n+\xi k^2) \end{bmatrix} \tag{20}$$

where

$$\Delta \equiv \alpha_n\alpha_p + (\alpha_p\xi - \beta^2)k^2 . \tag{21}$$

From (20), the fundamental four spectra are obtained as

$$E_n(k) = \frac{\pi\alpha_p k}{\Delta}, \qquad E_p(k) = \frac{\pi k(\alpha_n + \xi k^2)}{\Delta}, \qquad (22),\ (23)$$

$$H(k) = \frac{\pi\beta k^3}{\Delta}, \qquad S(k) = k^2 E_n(k). \qquad (24),\ (25)$$

By integrating (22)-(25) over k from k_{min} to k_{max}, it is possible to express the four fundamental invariants $<E_n>$, $<Ep>$, $<H>$, and $<S>$ in terms of α_n, α_p, β, ξ, k_{min} and k_{max}. Conversely, given k_{min} and k_{max}, $<E_n>$, $<Ep>$, $<H>$, and $<S>$ satisfying realizability conditions α_n, α_p, β and ξ are uniquely determined but cannot be obtained in closed analytic form.

As a special case of interest, notice that β=0 is equivalent to $<H>$=0. In this case, E_n and E_p reduce to the absolute equilibrium spectra for two-dimensional turbulence in horizontal plane and for two-dimensional passive scalar, respectively, as

$$E_n(k) = \frac{\pi k}{\alpha_n + \xi k^2}, \qquad E_p(k) = \frac{\pi k}{\alpha_p}. \qquad (26),\ (27)$$

For the above $En(k)$, the constants α_n and ξ can determine in terms of the mean kinetic energy and enstrophy. If α_n and ξ are positive, the wave number $(\alpha_n/\xi)^{1/2}$ can easily be shown to be of the order of the wave number

$$k_i = \left(\frac{\int_0^\infty k^2 E(k)\mathrm{d}k}{\int_0^\infty E(k)\mathrm{d}k} \right)^{1/2} \tag{28}$$

characteristic of an average wave number of the spectrum[17] for $k >> k_i$ and k-energy equipartition spectrum for $k \to 0$.

The possibility of a negative α_n (with a positive ξ) is permitted if there exists a lower wave number bounded k_{min} such that

$$k_{min} = \left(\frac{-\alpha_n}{\xi}\right)^{1/2}. \tag{29}$$

Then if k_{min} is very close to $(-\alpha_n/\xi)^{1/2}$, one can obtain an arbitrarily high kinetic energy spectrum in the vicinity of k_{min}. This is what Kraichnan called "negative temperature states", since the "temperature" of the system is characterized by α_n, which is here negative. Such a behavior could of course be considered as an indication of the inverse energy cascade in the viscous problem.

Let us now consider the two-dimensional passive scalar: since it possesses only one quadratic invariant (the scalar variance), the same analysis will lead to equipartition of scalar variance among the wave vectors **k**, and then to a scalar spectrum proportional to k. If one is in the case where α_n is positive, the scalar spectrum and the kinetic energy spectra will be both proportional to k in low wave numbers, while the enstrophy spectrum will be k^3. Consequently, there will be more scalar than enstrophy in this infrared spectral range: such a behaviour was verified by Holloway, *et al.*[18] with an inviscid truncated direct numerical simulation, and shows the dynamical difference of the scalar and the enstrophy in this range.

It is true that the absolute equilibrium ensembles are very far from the actual non-equilibrium state of turbulence. Nevertheless their value may be in showing the direction in which the actual states may plausibly be expected to transfer excitation, since within any given wavenumber band the characteristic time for reaching absolute equilibrium is the same as the eddy turnover time. For purely two-dimensional turbulence, the prediction by absolute-equilibrium ensembles of an inverse energy cascade has, in fact, been verified by numerical experiment.[19]

Let us here examine the three absolute equilibrium spectra $E_n(k)$, $E_p(k)$ and $H(k)$ as in (22)-(24), respectively.

(1) When $\alpha_p\xi-\beta^2>0$, $E_n(k)$ can have negative temperature, i.e. $\alpha_n<0$. Then, in the viscous problem, $E_n(k)$ would be transferred to low wave numbers, like the inverse cascade in two-dimensional turbulence.

(2) Contrary, when $\alpha_p\xi-\beta^2<0$, the inverse cascade of $E_n(k)$ is completely inhibited in the presence of helicity. However, it is a question whether $\alpha_p\xi-\beta^2<0$ has a reality.

(3) $E_p(k)$ in (23) suggests that if helicity takes some special values, it is possible that $E_p(k)$ in the viscous flow evolves toward low wave numbers like the inverse cascade.

(4) At large k, $H(k)$ and $E_p(k)$ can be related as

$$H(k) \approx \frac{\mu}{\beta} E_p(k) \tag{30}$$

This indicates that parallel components of the flow receive influences from helicity much more than transverse components.

III. ENERGY SPECTRUM EQUATIONS

A. Basic equation

Under a uniform magnetic field $\mathbf{B}_0$, the dynamics of low magnetic Reynolds number flows such as liquid metals is described by so called quasistatic approximation. The rotational part of randomly stirred external force is denoted by $\mathbf{f}(\mathbf{x},t)$. The spectral equations of fluid motion are then given by [4]

$$\left[\frac{\partial}{\partial t}+\nu k^2+\frac{\sigma}{\rho k^2}(\mathbf{B}_0\cdot\mathbf{k})^2\right]u_a(\mathbf{k},t)=-ik_a\frac{P(\mathbf{k},t)}{\rho}-ik_m\int u_a(\mathbf{k}-\mathbf{p},t)u_m(\mathbf{p},t)d\mathbf{p}+f_a(\mathbf{k},t), \tag{31}$$

$$k_a u_a(\mathbf{k},t)=0 \tag{32}$$

where ν is the kinematic viscosity, σ the electric conductivity, ρ the uniform fluid density; $P(\mathbf{k},t)$ is the spectral pressure; $f_a(\mathbf{k},t)$ is the Fourier transform of $f_a(\mathbf{x},t)$. Note that the electromagnetic effect reduces to an anisotropic Joule dissipation with a rate proportional to the squared cosine of the angle between the wave vector $\mathbf{k}$ and the uniform external magnetic field $\mathbf{B}_0$, but independent of wavenumber.

B. The equation for energy spectrum in homogeneous axisymmetric helical turbulence[1,2]

In Fourier space, the second-order velocity correlation in homogeneous turbulence is given by

$$S_{ab}(\mathbf{k},t)=\langle u_a(\mathbf{k},t)u_b^*(\mathbf{k},t)\rangle \tag{33}$$

The equation for $S_{ab}(k,t)$ is derived straightforwardly from (31) as

$$\left[\frac{\partial}{\partial t}+2\left(k^2+\frac{\sigma}{\rho}B_0^2\mu^2\right)\right]S_{ab}(\mathbf{k},t)=T_{ab}(\mathbf{k},t)+W_{ab}(\mathbf{k},t) \tag{34}$$

where μ is the cosine of the angle between $\mathbf{k}$ and $\mathbf{B}_0$: $T_{ab}(\mathbf{k},t)$ and $W_{ab}(\mathbf{k},t)$ denote spectral tensor for energy transfer and input energy due to randomly stirred external forces, respectively. The energy transfer term $T_{ab}(\mathbf{k},t)$ consists of two parts. One is the pressure-velocity correlation $\Phi_{ab}(\mathbf{k},t)$ and the other is the inertial transfer $\Gamma_{ab}(\mathbf{k},t)$ as

$$T_{ab}(\mathbf{k},t)=\phi_{ab}(\mathbf{k},t)+\Gamma_{ab}(\mathbf{k},t), \tag{35}$$

$$\phi_{ab}(\mathbf{k},t)=-\frac{i}{\rho}\left[k_a\langle p(\mathbf{k},t)u_b^*(\mathbf{k},t)\rangle-k_b\langle p^*(\mathbf{k},t)u_a(\mathbf{k},t)\rangle\right], \tag{36}$$

$$\Gamma_{ab}(\mathbf{k},t)=\frac{1}{2}k_m\int \mathrm{d}\mathbf{p}\left[\left\langle u_m(\mathbf{k}-\mathbf{p},t)u_a(\mathbf{p},t)u_b^*(\mathbf{k},t)\right\rangle - \left\langle u_m^*(\mathbf{k}-\mathbf{p},t)u_a^*(\mathbf{p},t)u_b(\mathbf{k},t)\right\rangle\right]. \tag{37}$$

Note that in the incompressible constraint (32), $P(\mathbf{k},t)$ is expressed as

$$P(\mathbf{k},t)=-i\,\rho\frac{k_l k_m}{k^2}\int \mathrm{d}\mathbf{p}\left\langle u_l(\mathbf{k}-\mathbf{p},t)u_m(\mathbf{p},t)\right\rangle \tag{38}$$

In incompressible homogeneous axisymmetric helical turbulence, the tensor function of rank 2 such as $S_{ab}(\mathbf{k},t)$ and $W_{ab}(\mathbf{k},t)$ is expressed with three scalar functions. In the coordinates that the direction of $\mathbf{B}_0$ is the x_3-axis, the energy spectrum tensor $S_{ab}(\mathbf{k},t)$, for instance, is written as

$$S_{ab}(\mathbf{k},t)=\langle u_a(\mathbf{k},t)u_b^*(\mathbf{k},t)\rangle$$

$$=\frac{1}{4\pi k^2}\left[\Delta_{ab}(\mathbf{k})E(k,\mu,t)+\Delta_{a3}(\mathbf{k})\Delta_{b3}(\mathbf{k})F(k,\mu,t)\right]+\frac{i}{4\pi k^4}\varepsilon_{abs}k_s H(k,\mu,t). \tag{39}$$

where

$$\Delta_{ab}(\mathbf{k})=\delta_{ab}-\frac{k_a k_b}{k^2}, \tag{40}$$

and ε_{abs} is the fundamental antisymmetric tensor of rank 3. Total energy of the velocity components perpendicular and parallel to $\mathbf{B}_0$ are then given as

$$E_{aa}(t)=\int_0^\infty dk\int_0^1 d\mu\hat{E}_{aa}(k,\mu,t) \qquad (a=1\text{ or }3) \tag{41}$$

where

$$\hat{E}_{11}(k,\mu,t)=\hat{E}_{22}(k,\mu,t)=\frac{1}{2}4\pi k^2 S_{11}(\mathbf{k},t)=\frac{1}{4}\left\{E(k,\mu,t)+\mu^2\left[E(k,\mu,t)+(1-\mu^2)F(k,\mu,t)\right]\right\}, \tag{42}$$

$$\hat{E}_{33}(k,\mu,t)=\frac{1}{2}4\pi k^2 S_{33}(\mathbf{k},t)=\frac{1}{2}(1-\mu^2)\left[E(k,\mu,t)+(1-\mu^2)F(k,\mu,t)\right]. \tag{43}$$

And total helicity is given by

$$H(t)=\frac{1}{2}\int i\varepsilon_{amb}k_m S_{ab}(\mathbf{k},t)\mathrm{d}\mathbf{k}=\frac{1}{2}\int_0^\infty 4\pi k^2\mathrm{d}k\int_0^1 \mathrm{d}\mu\; i\,\varepsilon_{amb}k_m i\varepsilon_{abs}k_s\frac{H(k,\mu,t)}{4\pi k^4}$$

$$=\int_0^\infty\int_0^1 H(k,\mu,t)\mathrm{d}k\mathrm{d}\mu \tag{44}$$

To obtain a closed system of $S_{ab}(\mathbf{k},t)$, we use the EDQNM technique. Substituting (39) into (34), we obtain the equations for the spectra $E(k,\mu,t)$, $F(k,\mu,t)$ and $H(k,\mu,t)$.

Taking the summation of $S_{aa}(\mathbf{k},t)$ in (34), we have

$$\left[\frac{\partial}{\partial t}+2(\nu k^2+N_0\mu^2)\right]\left[2E(k,\mu,t)+(1-\mu^2)F(k,\mu,t)\right]=T_{ii}(k,\mu,t), \tag{45}$$

$$\begin{aligned}T_{ii}(k,\mu,t)=\frac{1}{2\pi}\int\int_{\Delta/2}\frac{\mathrm{d}p\,\mathrm{d}q}{pq}\int_{-\pi/2}^{\pi/2}\mathrm{d}\varphi\theta_{kpq}(t)\\ \times\Big\{k^3\{[J_1E(q,\mu'',t)+G_1F(q,\mu'',t)]E(p,\mu',t)+[G_1'E(q,\mu'',t)+A_1F(q,\mu'',t)]F(p,\mu',t)\}\\ +\{p^3[M_1E(q,\mu'',t)+Q_1F(q,\mu'',t)]+q^3[M_1'E(p,\mu',t)+Q_1'F(p,\mu',t)]\}E(k,\mu,t)\\ +\{p^3[M_3E(q,\mu'',t)+Q_3F(q,\mu'',t)]+q^3[M_3'E(p,\mu',t)+Q_3'F(p,\mu',t)]\}F(k,\mu,t)\\ +kC_{11}H(q,\mu'',t)H(p,\mu',t)+[pC_{12}H(q,\mu'',t)+qC_{12}'H(p,\mu',t)]H(k,\mu,t)\Big\}\end{aligned} \tag{46}$$

Geometrical factors J_1, G_1, G_1', A_1, M_1, M_1', Q_1, Q_1' M_3, M_3', Q_3, and Q_3' are the same as those in our previous paper[1], and

$$C_{11}=-2\frac{k^2}{pq}(x+yz)\ , \tag{47}$$

$$C_{12}=2\frac{p^2}{qk}z(y+xz)\ . \tag{48}$$

The variables μ' *and* μ'' are the cosine of the angles between $\mathbf{p}$ and $\mathbf{B}_0$, and $\mathbf{q}$ and $\mathbf{B}_0$, respectively. They are related with integral variable ϕ as

$$\mu'=-\mu z-\sqrt{(1-\mu^2)(1-z^2)}\ \sin\varphi, \tag{49}$$

$$\mu''=-\mu y+\sqrt{(1-\mu^2)(1-y^2)}\ \sin\varphi. \tag{50}$$

The variables x, y and z are expressed in terms of k, p and q as

$$x=-\frac{(\mathbf{p}\cdot\mathbf{q})}{pq}=\frac{(p^2+q^2-k^2)}{2pq}, \tag{51}$$

$$y=-\frac{(\mathbf{q}\cdot\mathbf{k})}{qk}=\frac{(q^2+k^2-p^2)}{2qk}, \tag{52}$$

$$z=-\frac{(\mathbf{k}\cdot\mathbf{p})}{kp}=\frac{(k^2+p^2-q^2)}{2kp}. \tag{53}$$

In the EDQNM theory, $\theta_{kpq}(t)$ is given as

$$\theta_{kpq}(t) = \frac{1 - \exp\left\{-\left[\nu(k^2 + p^2 + q^2) + \left(\frac{\sigma}{\rho}\right)B_0^2(\mu^2 + \mu'^2 + \mu''^2) + \gamma_e(k) + \gamma_e(p) + \gamma_e(q)\right]\right\}}{\left[\nu(k^2 + p^2 + q^2) + \left(\frac{\sigma}{\rho}\right)B_0^2(\mu^2 + \mu'^2 + \mu''^2) + \gamma_e(k) + \gamma_e(p) + \gamma_e(q)\right]} \tag{54}$$

where $\gamma_e(k)$ is the eddy damping factor , given by

$$\gamma_e(k) = \lambda \left(4\pi \int_0^k \int_0^1 k^2 S_{aa}(\mathbf{k},t)\mathrm{d}k\,\mathrm{d}\mu\right)^{1/2}. \tag{55}$$

We adopted here the numerical factor λ as

$$\lambda = 0.225. \tag{56}$$

The value 0.225 has been adopted by Lesuier *et al.* in the study of isotropic helical turbulence.

The p-q integral domain $\mathbf{\Delta}$ is given under the condition as

$$|z| \leq 1. \tag{57}$$

Considering $S_{33}(\mathbf{k},t)$, we have

$$\left[\frac{\partial}{\partial t} + 2(\nu k^2 + N_0\mu^2)\right]\left[E(k,\mu,t) + (1-\mu^2)F(k,\mu,t)\right] = \frac{1}{(1-\mu^2)}T_{33}(k,\mu,t) \tag{58}$$

where

$$\begin{aligned}
T_{33}(k,\mu,t) = \frac{1}{2\pi}\int\int_{\Delta/2} \frac{\mathrm{d}p\,\mathrm{d}q}{pq}\int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi\theta_{kpq}(t) \\
\times\Big\{k^3\left\{\left[J_3E(q,\mu'',t)+G_3F(q,\mu'',t)\right]E(p,\mu',t)+\left[G_3'E(q,\mu'',t)+A_3F(q,\mu'',t)\right]F(p,\mu',t)\right\} \\
+\left\{p^3\left[M_3E(q,\mu'',t)+Q_3F(q,\mu'',t)\right]+q^3\left[M_3'E(p,\mu',t)+Q_3'F(p,\mu',t)\right]\right\}\left[E(k,\mu,t)+(1-\mu^2)F(k,\mu,t)\right] \\
+kC_{31}H(q,\mu'',t)H(p,\mu',t)+\left[pC_{32}H(q,\mu'',t)+qC_{32}'H(p,\mu',t)\right]H(k,\mu,t)\Big\}
\end{aligned} \tag{59}$$

Geometrical factors are

$$C_{31} = -2\frac{k^2}{pq}\left[(x+yz)-\mu^2x+\mu(\mu'y+\mu''z)+\mu'\mu''\right], \tag{60}$$

$$C_{32} = 2\frac{p^2}{qk}\left[(1-\mu^2)z(y+xz)-\mu''(x+2yz)-(1-2z^2)\mu''(\mu'+\mu z)-\mu'\mu''(y+2xz+2yz^2)\right]. \tag{61}$$

For $H(k,\mu,t)$, we have

$$\left[\frac{\partial}{\partial t} + 2(\nu k^2 + N_0\mu^2)\right]H(k,\mu,t) = T_H(k,\mu,t) \tag{62}$$

where

$$T_H(k,\mu,t)=\frac{1}{2\pi}\int\int_{\Delta/2}\frac{\mathrm{d}p\,\mathrm{d}q}{pq}\int_{-\pi/2}^{\pi/2}\mathrm{d}\varphi\theta_{kpq}(t)$$
$$\times\{k^3[D_{11}E(p,\mu't)+V_{11}F(p,\mu',t)]H(q,\mu'',t)+k^3[D_{11}'E(q,\mu''t)+V_{11}'F(q,\mu'',t)]H(p,\mu',t)$$
$$+p^3[D_{21}E(q,\mu''t)+V_{21}F(q,\mu'',t)]H(k,\mu,t)+q^3[D_{21}'E(p,\mu't)+V_{21}'F(p,\mu',t)]H(k,\mu,t)$$
$$+p^3[D_{22}E(k,\mu,t)+V_{22}F(k,\mu,t)]H(q,\mu'',t)+q^3[D_{22}'E(k,\mu,t)+V_{22}'F(k,\mu,t)]H(p,\mu',t)\}. \tag{63}$$

The geometrical factors are

$$D_{11}=\frac{k}{q}(y-xz-2yz^2), \tag{64}$$
$$V_{11}=\frac{k}{q}(\mu+\mu'z)[(\mu''+\mu'x)+2y(\mu+\mu'z)], \tag{65}$$
$$D_{21}=-(xy+z^3), \tag{66}$$
$$V_{21}=[(\mu^2+\mu'^2)z+\mu\mu'(1+2z^2)+\mu\mu''(x+2yz+2xz^2)+\mu'\mu''(y+2xz+2yz^2)+\mu''^2(xy+z-z^3)], \tag{67}$$
$$D_{22}=\frac{k}{q}z(y+xz), \tag{68}$$
$$V_{22}=\frac{k}{q}(\mu'+\mu z)[(\mu''+\mu y)+x(\mu'+\mu z)+2z(\mu'y-\mu''z)]. \tag{69}$$

In the limit of two-dimensional turbulence, energy spectra $E(k,\mu,t)$ and $F(k,\mu,t)$, and helicity spectrum $H(k,\mu,t)$ are expressed with those at μ=0. In this case, the energy spectrum for the velocity components transverse to $\mathbf{B}_0$ is expressed with $E(k,\mu,t)=E(k,t)\delta(\mu)$ that is the same as that in two-dimensional isotropic turbulence, and the energy spectrum for the parallel velocity components is expressed with $[E(k,\mu=0,t)+F(k,\mu=0,t)]/2$ that is the same as the equation for two-dimensional passive scalar. Purely two-dimensional turbulence is then expressed with $E(k,\mu=0,t)$ and $F(k,\mu=0,t)=-E(k,\mu=0,t)$. The expression of three-dimensional isotropic turbulence is given by the energy spectrum function $E(k,\mu,t)$ that is independent of μ, and $F(k,\mu,t)=0$.

IV. RESULTS

A. Numerical calculation

The equations for $E(k,\mu,t)$, $F(k,\mu,t)$ and $H(k,\mu,t)$ are solved numerically. The initial turbulent field is assumed to be isotropic. We initiate the computation in such a way that the initial isotropic turbulence is freely decayed at first until t=4. At t=4, a uniform external magnetic field $\mathbf{B}_0$ and randomly stirred external forces are imposed simultaneously.

The initial energy spectrum for the isotropic turbulence is assumed as

$$E(k,\mu,t)=16\left(\frac{2}{\pi}\right)^{1/2}u_0^2\frac{k^4}{k_0^5}\exp\left(-\frac{2k^2}{k_0^2}\right), \tag{70}$$

$$F(k,\mu,t) = H(k,\mu,t)=0 \tag{71}$$

where u_0 is the initial root-mean-square (rms) velocity and k_0 is the wave number at which the initial energy spectrum takes its maximum. The wavenumber k and the time t are non-dimensionalized with k_0 and $t_0=1/(k_0u_0)$, respectively. The initial Reynolds number and the initial interaction number are defined, respectively, by

$$R_0=\frac{u_0}{\nu k_0}, \qquad N_0=\frac{1}{\rho_0}\sigma B_0^2 t_0. \tag{72), (73}$$

The strength in the Joule dissipation relative to the nonlinear effect is measured by the interaction parameter.

The randomly stirred external forces applied to turbulence are assumed to be stationary as well as homogeneous and axisymmetric. In this case the input energy spectrum $W_{ab}(\mathbf{k})$ due to random external forces would be given as

$$W_{ab}(\mathbf{k})=\frac{1}{4\pi k^2}\left[\Delta_{ab}(\mathbf{k})W_1(k,\mu)+\Delta_{a3}(\mathbf{k})\Delta_{b3}(\mathbf{k})W_2(k,\mu)\right]++\frac{i\,\varepsilon_{abs}k_s}{4\pi k^4}W_3(k,\mu). \tag{74}$$

The energy spectra for random external forces perpendicular and parallel to $\mathbf{B}_0$ are written as

$$\widehat{W}_{aa}(k)=\int_0^1 \mathrm{d}\mu 4\pi k^2 W_{aa}(\mathbf{k}) \tag{75}$$

where

$$\widehat{W}_{11}(k,\mu)=\widehat{W}_{22}(k,\mu)=\frac{1}{2}4\pi k^2 W_{11}(\mathbf{k})=\frac{1}{4}\left\{W_1(k,\mu)+\mu^2\left[W_1(k,\mu)+(1-\mu^2)W_2(k,\mu,t)\right]\right\}, \tag{76}$$

$$\widehat{W}_{33}(k,\mu)=\frac{1}{4}4\pi k^2 W_{33}(\mathbf{k})=\frac{1}{2}(1-\mu^2)[W_1(k,\mu)+(1-\mu^2)W_2(k,\mu)]. \tag{77}$$

And the helicity spectrum due to randomly stirred external forces is given by

$$W_H(k)=\frac{1}{2}\int_0^1 \mathrm{d}\mu\; i\;\varepsilon_{imj}k_m 4\pi k^2 W_{ij}(\mathbf{k})=\int_0^1 \mathrm{d}\mu W_3(k,\mu)\;. \tag{78}$$

In numerical calculations, we divide the wavenumber k as

$$k_{n-m}=2^{(n-m)/4}, \qquad n=1,2,\ldots n_f \tag{79}$$

where m is fixed at 27 and $n_f=59$. The random external forces are assumed to be quasi-two-dimensional. Further, we assume that the energy due to the quasi-two-dimensional random forces is fed in a narrow band of wave numbers as

$$W_1(k,\mu=0)=0.5 \;\; and \;\; W_3(k,\mu=0)=kW_1(k,\mu=0) \;\; at \;\; k_{-1}=2^{-1/4}, k_0=1, k_1=2^{1/4}. \tag{80}$$

The range of angular variable μ between 0 and 1 is divided into five equal intervals, i.e. the mesh points of μ are given by

$$\mu_m = \frac{m}{5} \tag{81}$$

with $m=0,1,\ldots,5$. The p-q integrations in the pressure-velocity correlation and the inertial transfer terms are made following Leith,[20,21] and the μ-integration is carried out by using Simpson's rule. Time integral is made by a simple difference scheme with the time step $\Delta t=0.004$.

B. Anisotropy strength

Anisotropy strength in turbulence increases with $\mathbf{B}_0$. As a measure of it, we introduce the angles $\alpha_{aa}(t)$ $(a=1$ or $3)$, defined by

$$\cos^2\alpha_{aa}(t) = \frac{1}{E_{aa}(t)}\int_0^{\infty}\int_0^1 \mu^2\,\hat{E}_{aa}(k,\mu,t)\mathrm{d}k\mathrm{d}\mu \qquad (a = 1 \text{ or } 3). \tag{82}$$

where $\hat{E}_{11}(k,\mu,t)$ and $\hat{E}_{33}(k,\mu,t)$ are given as in (42) and (43).

This sort of angle was first introduced by Moreau[22] to characterize the elongation in the direction of the external field of the eddies still containing energy at time t. Most of the Joule dissipation takes place in the cone of the axis $\mathbf{B}_0$ and the semi-angle $\alpha_{11}(t)$ and $\alpha_{33}(t)$.

In Fig.1, we show the plots of $\cos^2\alpha_{aa}(t)$ in cases 1 and 2. In both cases, $\cos^2\alpha_{11}(t)$ asymptote to almost the same constant of small 4×10^{-3}, meaning that excitations in the turbulence are mostly confined in a small region between the angles [1.507, 1.571(=$\pi/2$)]. Thus, two dimensionality in turbulence is fairly strong. We can expect then that in horizontal plane, the turbulence in both cases behave like a two-dimensional isotropic turbulence, and in the direction of $\mathbf{B}_0$, they behave like a two-dimensional passive scalar.

C. Evolution of energy and helicity spectra

Figure 2 shows the evolution of $E_{11}(k,t)$ and $E_{33}(k,t)$ in case 1, and Fig.3 shows $E_{11}(k,t)$, $E_{33}(k,t)$ and $H(k,t)$ in case 2. The energy and helicity spectra are expressed in terms of $E(k,\mu,t)$, $F(k,\mu,t)$and $H(k,\mu,t)$ as

$$E_{aa}(k,t) = \int_0^1 \hat{E}_{aa}(k,\mu,t)\mathrm{d}\mu \qquad (a=1 \text{ or } 3), \tag{83}$$

$$H(k,t) = \int_0^1 4\pi k^2 i\, k_m \varepsilon_{amb} S_{ab}(\mathbf{k},t)\mathrm{d}\mu = \int_0^1 H(k,\mu,t)\mathrm{d}\mu\,. \tag{84}$$

where $\hat{E}_{11}(k,\mu,t)$, $\hat{E}_{33}((k,\mu,t)$ and $H(k,\mu,t)$ are given as in (42)-(44).

In a study of helical isotropic turbulence, Kraichnan[8] has shown that helicity prevents excitations from transferring to high and low wave numbers, and pointed out that in isotropic turbulence the inhibition of transfer to high wavenumber is not interest but may be important in other situation such as anisotropic turbulence.

In horizontal plane, $E_{11}(k,t)$ in both cases evolve to low wave numbers, like the inverse energy cascade in two-dimensional turbulence. However, the significant difference between cases 1 and 2 is not observed, although in the transfer toward high wavenumber, the difference is apparently observed (Fig.4). Thus, in the evolution of $E_{11}(k,t)$,we cannot consider that helicity plays a significant role to prevent excitations from transferring to low wavenumber. The inverse cascade is known to come from the fact that enstrophy is a constant of motion in two-dimensional turbulence. From this, we can suppose that in strongly anisotropic turbulence, the influences of enstrophy becomes strong much more than that of helicity so that the effects of helicity virtually disappear.

Meanwhile, in the evolution of $E_{33}(k,t)$, it seems that a role of helicity is rather significant in preventing excitations from transferring to low wavenumber as well as to high wavenumber (Fig.5). Because of the inhibition of the transfer, the excitations that are generated by stirring the fluid with external random forces and simultaneously provided with angular transfers from the horizontal plane to the parallel direction to $\mathbf{B}_0$ are accumulated in a rather narrow band of medium wavenumbers. As a result, the peak of $E_{33}(k,t)$ in helical case 2 increases with time, moving to low wavenumber as that of $E_{11}(k,t)$ does. Thus in strongly anisotropic helical turbulence, $E_{33}(k,t)$ can evolve like an inverse cascade: that is, it appears as if an ordered motion is produced in $E_{33}(k,t)$.

As time goes on, it is observed that $H(k,t)$ and $E_{33}(k,t)$ in helical case 2 come to resemble each other. This can be understood from the fact that in the limit of two-dimensional turbulence, vorticity and velocity components in the direction of $\mathbf{B}_0$ follow the same equation as that of two-dimensional passive scalar. Figures 4 and 5 show that the effect of helicity in turbulence decreases rather rapidly when small-scale components of helicity approach an equilibrium between external random forces and molecular dissipation at high wavenumber. Apparently this corresponds to the following fact obtained by Andre *et al.*[12] in a study of decaying helical isotropic turbulence. That is, " In decaying turbulence, the enstrophy of helicity $\int_0^\infty k^2 H(k,t)\,dk$ blows up at the time same as that of energy. At that time, the kinetic energy and helicity start being dissipated at a finite rate, and the helicity spectrum follows a linear cascade in the inertial range as $H(k,t) \sim (\varepsilon_H/\varepsilon)\, E(k,t)$. The relative helicity $H(k,t)/kE(k,t)$ is then proportional to k^{-1}, and decreases rapidly with k. Thus, after the time when the enstrophy of helicity blows up, helicity has no real influence on the energy flux."

D. Evolution of total energy and total helicity

We examine the evolution of total energy of the MHD turbulence when it is subjected to a uniform magnetic field $\boldsymbol{B}_0$ and randomly stirred forces. In the equation for the total energy of the turbulence, the inertial transfer term must satisfy the conservation law as

$$\int_0^\infty \int_0^1 \Gamma_{11}(k,t) = \int_0^\infty \int_0^1 \Gamma_{33}(k,t) = 0. \tag{85}$$

The equations for the total energy of the MHD turbulence [see (41)], perpendicular and parallel to $\mathbf{B}_0$, are then given, respectively, by

$$\frac{\partial}{\partial t}E_{11}(t) = -\varepsilon_{11}(t) - D_{11}(t) - \phi_{11}(t) + W_{11}(t), \tag{86}$$

$$\frac{\partial}{\partial t}E_{33}(t) = -\varepsilon_{33}(t) - D_{33}(t) + \phi_{33}(t) + W_{33}(t), \tag{87}$$

where angular transfer terms $\Phi_{aa}(t)$ ($a = 1$ or 3) must satisfy the conservation law as

$$\phi_{11}(t) = \frac{1}{2}\phi_{33}(t), \tag{88}$$

and $\varepsilon_{aa}(t)$, $D_{aa}(t)$ and W_{aa}, ($a=1$ or 3) are the molecular and Joule dissipation rates, respectively, defined by

$$\varepsilon_{aa}(t) = \frac{2}{R_0}\int_0^\infty k^2 E_{aa}(k,t)\,\mathrm{d}k, \tag{89}$$

$$D_{aa}(t) = 2N_0\int_0^\infty \mathrm{d}k\int_0^1 \mathrm{d}\mu\,\mu^2 \hat{E}_{aa}(k,\mu,t), \qquad W_{aa}(t) = \int_0^\infty W_{aa}(k)\mathrm{d}k\,. \qquad (90),\ (91)$$

The equation for total helicity in (44) is obtained from (62) as

$$\frac{\partial}{\partial t}H(t) = -\varepsilon_H(t) - D_H(t) + W_H(t) \tag{92}$$

where $\varepsilon_H(t)$, $D_H(t)$ *and* $W_H(t)$ are

$$\varepsilon_H(t) = \frac{2}{R_0}\int_0^\infty k^2 H(k,t)\,\mathrm{d}k, \qquad D_H(t) = 2N_0\int_0^\infty \mathrm{d}k\int_0^1 \mathrm{d}\mu\,\mu^2 H(k,\mu,t), \qquad (93),\ (94)$$

$$W_H(t) = \int_0^\infty W_H(k)\mathrm{d}k\,. \qquad [\textit{see}\ (78)] \tag{95}$$

Figure 6 shows the plot of the total energy $E_{aa}(t)$ (a=1 or 3) in non helical case 1, and Fig.7 shows the total energy and total helicity $H(t)$ in helical case 2. At the time t=4, at which a uniform magnetic field $\mathbf{B}_0$ and randomly stirred external forces are imposed, energy in initially isotropic turbulence decreases sharply with time, and two-dimensionality in turbulence proceeds quickly, owing to the linear anisotropic Joule dissipation (first stage in the processes). Around a turnover time, total energy of the turbulence in cases 1 and 2 both reach almost the same minimum values whose magnitude depends on $\mathbf{B}_0$. At that time, the energy begins to increase because nonlinear effect becomes important (second stage): that is, the effects of angular transfer and inertial transfer are not negligible at this time any more.

Before long, enstrophy of $E_{33}(t)$ and "enstrophy" of $H(t)$ blow up at almost the same time. At that time, $E_{33}(t)$ reaches the maximum: we see that in case 1, that time is t=12, and in case 2, t=20. This agrees with the result by Andre *et al.*[11,12] in decaying helical isotropic turbulence that the time at which the enstrophy in helical case blows up is increased by a factor 2, compared with that in non helical case.

Soon, $E_{33}(t)$, decreasing down slightly from the maximum, is settled in a constant when small eddies reach an equilibrium between molecular dissipation and inertial transfer toward small eddies, although large eddies are not in an equilibrium. We see that the asymptotic constant in case 2 is 3.5 that is 1.5 times larger than that in case 1.

Note that $E_{11}(t)$ in horizontal plane is still growing as a result of inverse energy cascade. However, as time goes on further, $E_{11}(t)$ would reach an equilibrium stationary state, as shown in our previous paper, between Joule dissipation and randomly stirred external forces, even in the limit of strong N_0.

V. CONCLUDING REMARKS

First, we examined the absolute statistical equilibrium of the equations for truncated quasi-two-dimensional turbulence. Especially, we examined the three absolute equilibrium spectra $E_n(k)$, $E_p(k)$ and $H(k)$ as in (22)-(24), respectively.

Next, we examined the effect of helicity with the energy spectrum equation that is closed by the EDQNM theory, assuming that turbulent field is homogeneous and axisymmetric. We presented the results of two calculations for strongly anisotropic turbulence, as a preliminary study. The flows in horizontal plane evolved toward low wave numbers, like the inverse energy cascade in two-dimensional turbulence. In the evolution of $E_{11}(k,t)$, however, it could not seem that helicity plays a significant role to prevent excitations from transferring to low wave number. It is consider that in strongly anisotropic turbulence, the influence of enstrophy becomes strong much more than that of helicity so that the influence of helicity virtually disappears.

In the flows in the direction of symmetry axis, there existed an tendency to make a coherent structure. Because of the inhibition of the transfers to low as well as high wavenumbers, the excitations that are generated by stirring the fluid with external random forces and simultaneously provided with angular transfers from the horizontal plane to the parallel direction to $\mathbf{B}_0$ are accumulated in a rather narrow band of medium wavenumbers. As a result, in strongly anisotropic helical turbulence, $E_{33}(k,t)$ evolves like an inverse cascade: that is, it appears as if an ordered motion is produced in $E_{33}(k,t)$.

For understanding the effects of helicity in anisotropic turbulence more deeply, it will be necessary to examine more examples of calculations for various N_0 and initial conditions.

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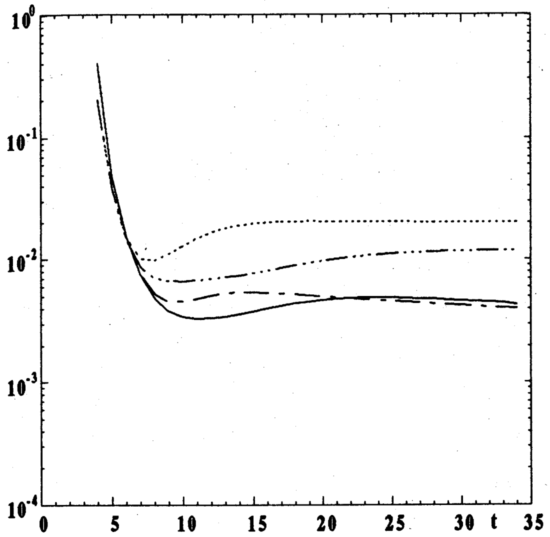

Fig. 1. (———), $\cos^2\alpha_{11}(t)$ in non helical case 1; (———), $\cos^2\alpha_{11}(t)$ in helical case 2;
(--------), $\cos^2\alpha_{33}(t)$ in non helical case 1; (——-—), $\cos^2\alpha_{33}(t)$ in helical case 2.

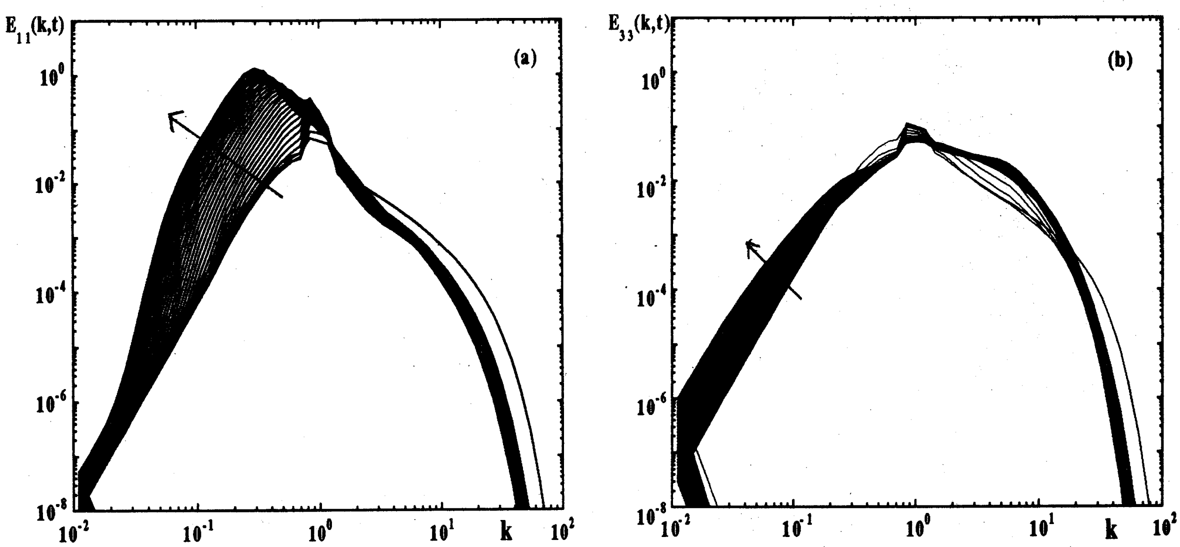

Fig.2. Evolution of energy spectra in non helical case 1. Arrows denote the direction of time.

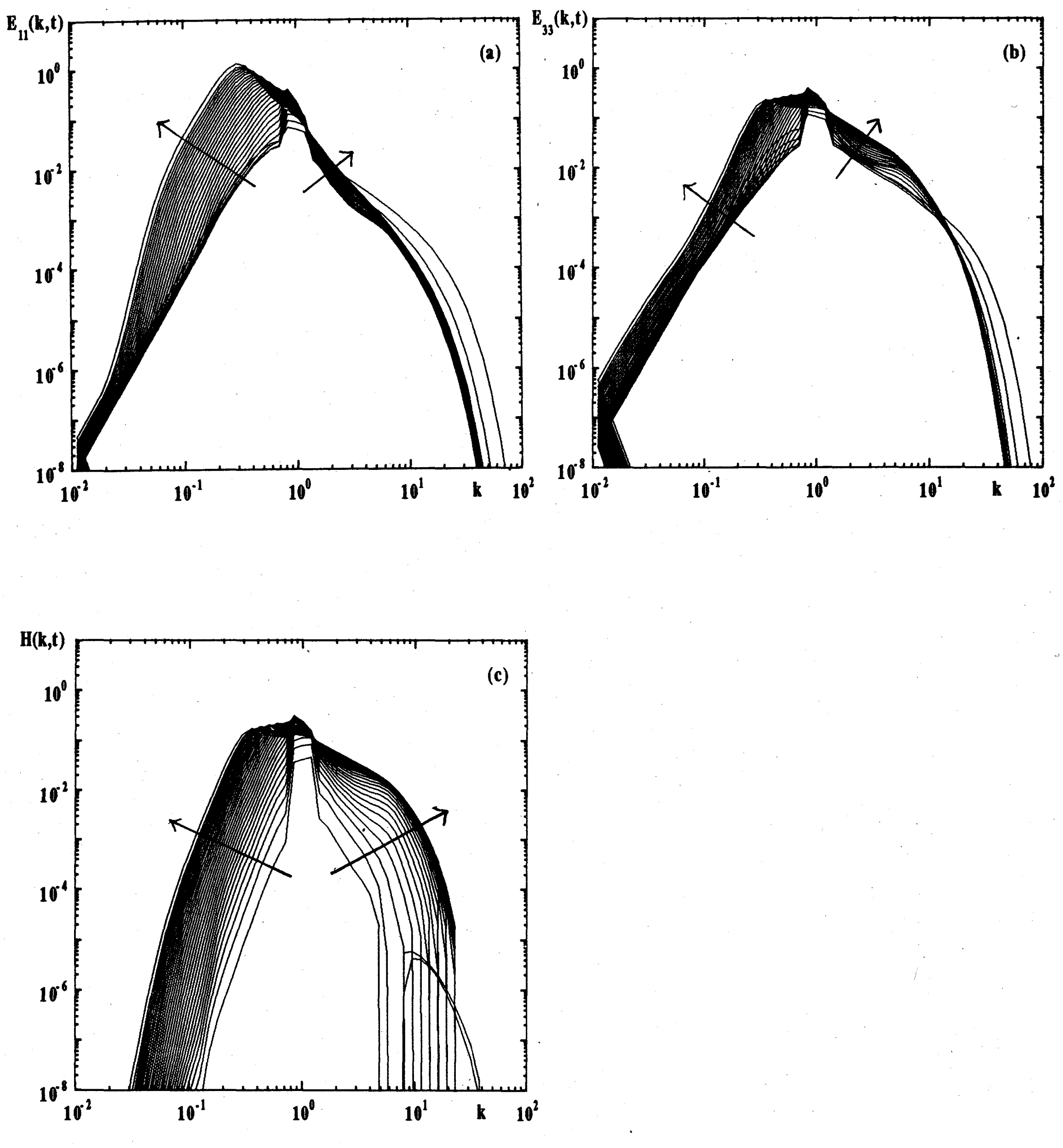

Fig.3. Evolution of energy and Helicity spectra in helical case 2. Arrows denote the direction of time.

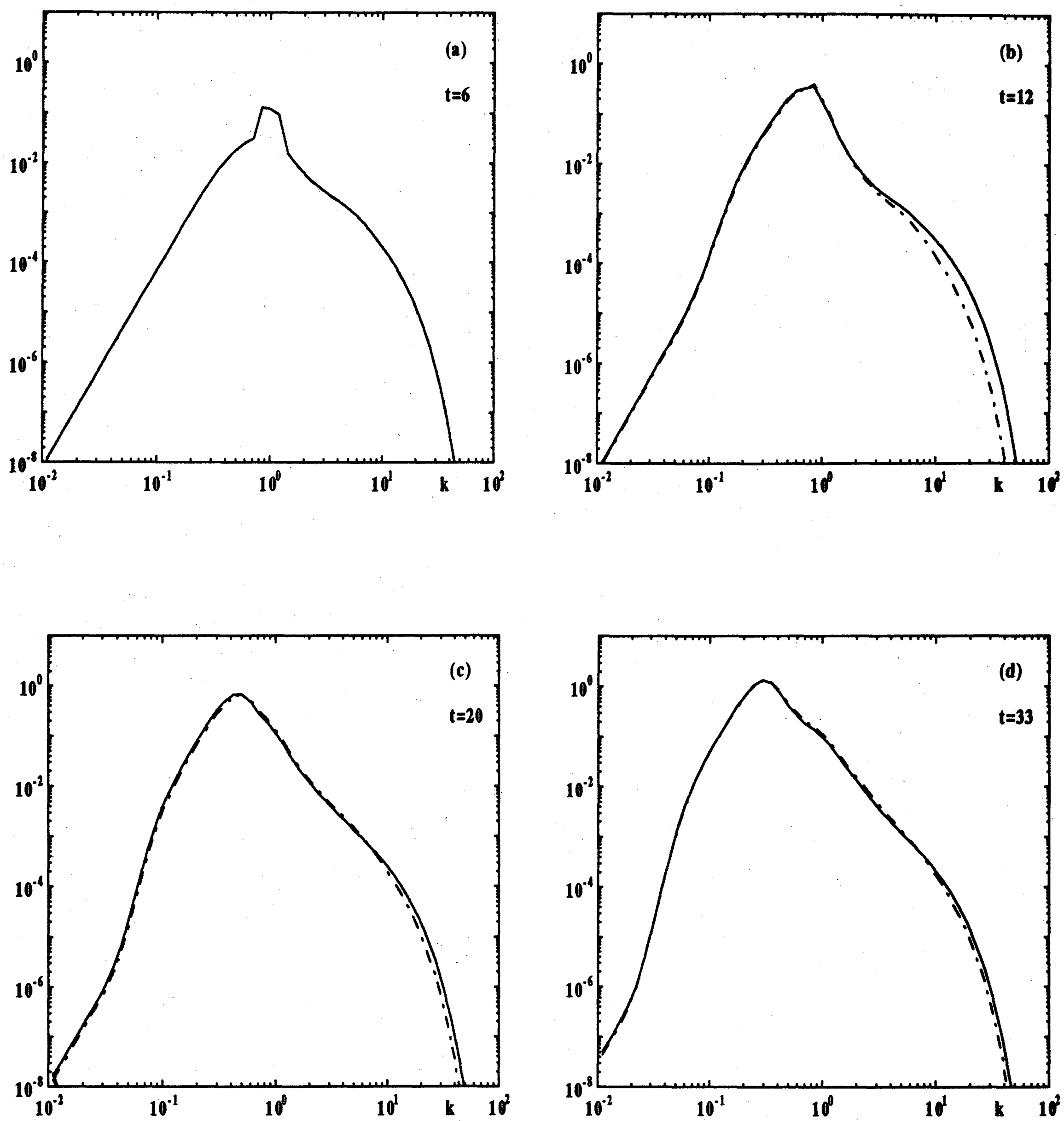

Fig.4. Energy spectrum $E_{11}(k,t)$: (--------), in non helical case 1; (———), in helical case 2.

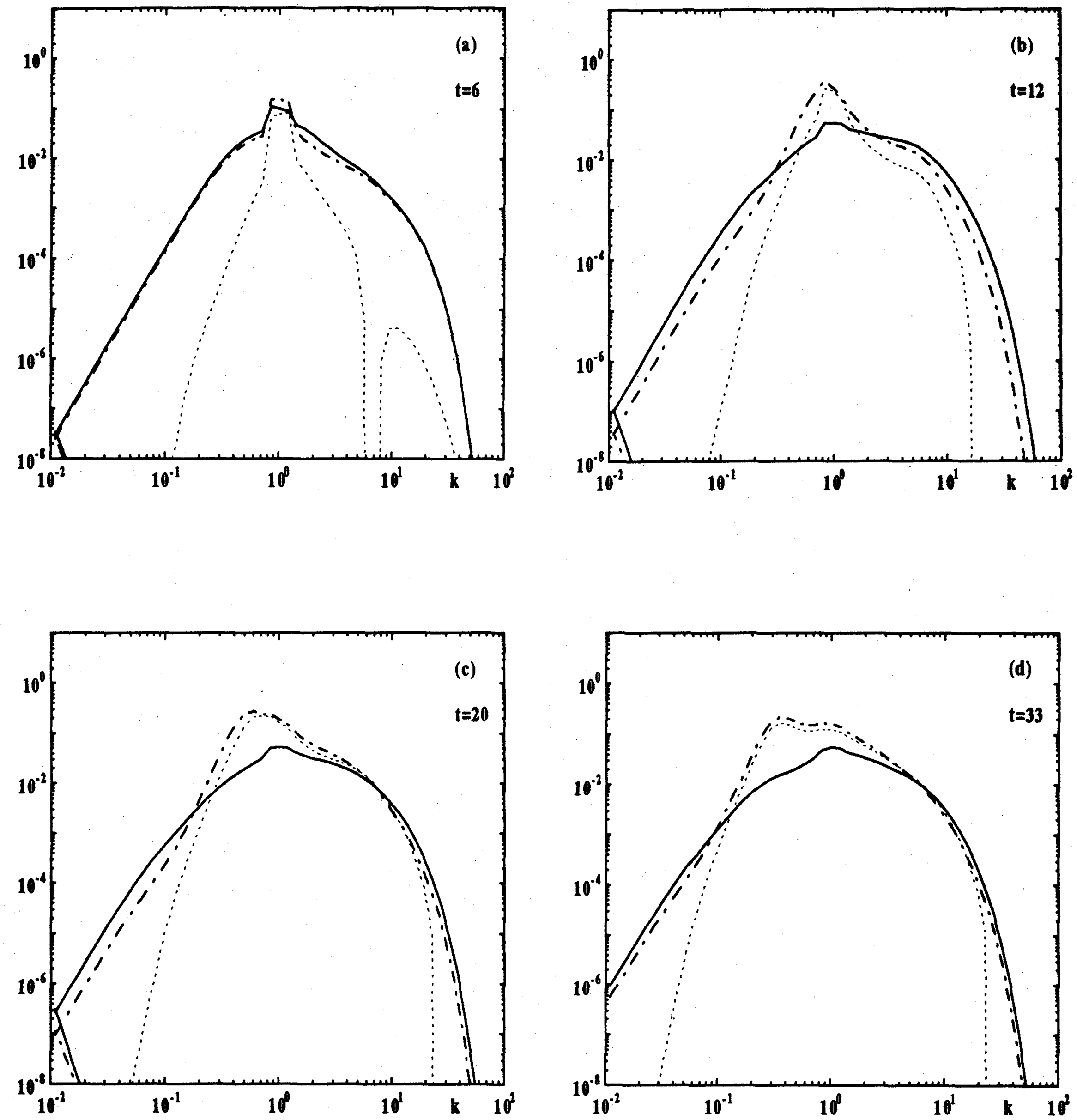

Fig.5. Energy spectrum $E_{33}(k,t)$and helicity spectrum $H(k,t)$: (———), $E_{33}(k,t)$ in non helical case 1; (— — —), $E_{33}(k,t)$ in helical case 2; (---------), $H(k,t)$ in case 2.

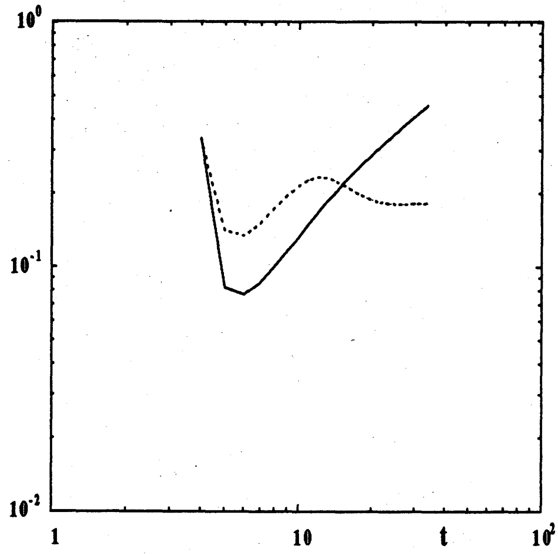

Fig.6. Total energy in non helical case 1: (———), $E_{11}(t)$; (--------), $E_{33}(t)$.

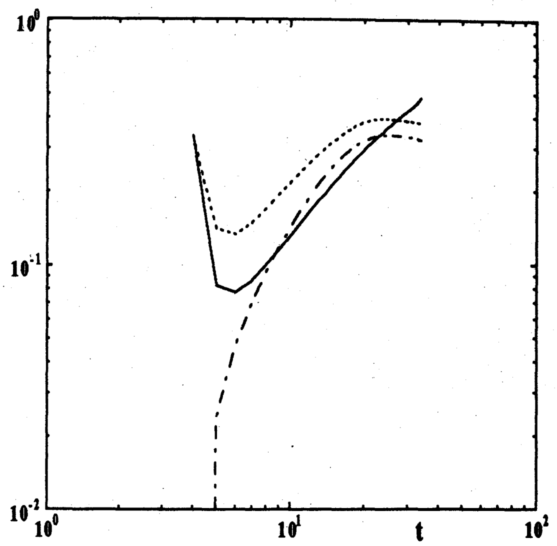

Fig.7. Total energy and helicity in helical case 2: (———),$E_{11}(t)$; (--------), $E_{33}(t)$; (—·—),$H(t)$.